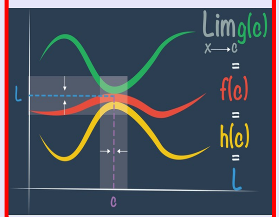
Calculus I

Lecture 3



Feb 19-8:47 AM

Evaluate

1) $\lim\limits_{x \to 0} \cos(x + \sin x) = \cos(0 + \sin 0)$

$= \cos(0 + 0) = \cos 0 = \boxed{1}$

2) $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x^2 + 2x - 3} = \dfrac{3^2 - 9}{3^2 + 2(3) - 3} = \dfrac{9 - 9}{9 + 6 - 3} = \dfrac{0}{12} = \boxed{0}$

$A^2 - B^2$

3) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x^3 - 8} = \dfrac{2^2 - 4}{2^3 - 8} = \dfrac{4 - 4}{8 - 8} = \dfrac{0}{0}$ I.F.

$A^3 - B^3$

$= \lim\limits_{x \to 2} \dfrac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}(x^2 + 2x + 4)} = \lim\limits_{x \to 2} \dfrac{x + 2}{x^2 + 2x + 4}$

$= \dfrac{2 + 2}{2^2 + 2(2) + 4} = \dfrac{4}{12} = \boxed{\dfrac{1}{3}}$

Jan 7-8:01 AM

4) $\lim_{h\to 0} \frac{(h-1)^3+1}{h} = \frac{(0-1)^3+1}{0} = \frac{(-1)^3+1}{0} = \frac{-1+1}{0} = \frac{0}{0}$ I.F.

$= \lim_{h\to 0} \frac{h^3-3h^2+3h-\cancel{1}\cancel{+1}}{h} = \lim_{h\to 0} \frac{\cancel{h}(h^2-3h+3)}{\cancel{h}}$

$= \lim_{h\to 0} (h^2-3h+3) = \boxed{3}$

5) $\lim_{x\to 16} \frac{4-\sqrt{x}}{x-16} = \frac{4-\sqrt{16}}{16-16} = \frac{4-4}{0} = \frac{0}{0}$ I.F.

$= \lim_{x\to 16} \frac{\overbrace{(4-\sqrt{x})}^{A-B}\overbrace{(4+\sqrt{x})}^{A+B}}{(x-16)(4+\sqrt{x})} = \lim_{x\to 16} \frac{\overbrace{4^2-(\sqrt{x})^2}^{A^2-B^2}}{(x-16)(4+\sqrt{x})}$

$= \lim_{x\to 16} \frac{\cancel{16-x}}{\cancel{(x-16)}(4+\sqrt{x})} = \lim_{x\to 16} \frac{-1}{4+\sqrt{x}} = \frac{-1}{4+\sqrt{16}} = \boxed{\frac{-1}{8}}$

Jan 7-8:10 AM

6) $\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{4+x} = \frac{\frac{1}{4}+\frac{1}{-4}}{4+(-4)} = \frac{\frac{1}{4}-\frac{1}{4}}{4-4} = \frac{0}{0}$ I.F.

LCD $= 4x$

$\lim_{x\to -4} \frac{4x(\frac{1}{4}+\frac{1}{x})}{4x(4+x)} = \lim_{x\to -4} \frac{\overset{1}{\cancel{x+4}}}{4x\cancel{(4+x)}} = \lim_{x\to -4} \frac{1}{4x} = \boxed{\frac{-1}{16}}$

7) $\lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \frac{\frac{1}{(x+0)^2}-\frac{1}{x^2}}{0} = \frac{\frac{1}{x^2}-\frac{1}{x^2}}{0} = \frac{0}{0}$ I.F.

LCD $= (x+h)^2 \cdot x^2$

$\lim_{h\to 0} \frac{(x+h)^2x^2\left(\frac{1}{(x+h)^2}-\frac{1}{x^2}\right)}{(x+h)^2x^2 \cdot h} = \lim_{h\to 0} \frac{x^2-(x+h)^2}{(x+h)^2x^2h}$

$= \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{(x+h)^2x^2h} = \lim_{h\to 0} \frac{-2xh-h^2}{(x+h)^2x^2h}$

$= \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{(x+h)^2 \cdot x^2 \cdot \cancel{h}} = \lim_{h\to 0} \frac{-2x-h^{\to 0}}{(x+h_{\to 0})^2 \cdot x^2}$

$= \frac{-2\cancel{x}}{x^2 \cdot \cancel{x^2}} = \boxed{\frac{-2}{x^3}}$

Jan 7-8:22 AM

8) $\lim\limits_{x \to -4} \dfrac{\sqrt{x^2+9} - 5}{x+4} = \cdots = \dfrac{0}{0}$ I.F.

$\underbrace{(\cancel{x+4})(x-4)}_{}$

$\underbrace{x^2-16}_{}$

$= \lim\limits_{x \to -4} \dfrac{(\sqrt{x^2+9}-5)(\sqrt{x^2+9}+5)}{(x+4)(\sqrt{x^2+9}+5)} = \lim\limits_{x \to -4} \dfrac{x^2+9-25}{(\cancel{x+4})(\sqrt{x^2+9}+5)}$

$= \lim\limits_{x \to -4} \dfrac{x-4}{\sqrt{x^2+9}+5} = \dfrac{-4-4}{\sqrt{(-4)^2+9}+5} = \dfrac{-8}{10} = \boxed{\dfrac{-4}{5}} = \boxed{-.8}$

9) If $4x-9 \le f(x) \le x^2-4x+7$ for $x \ge 0$, find $\lim\limits_{x \to 4} f(x)$

$\lim\limits_{x \to 4} (4x-9) = 7$

$\lim\limits_{x \to 4} (x^2-4x+7) = 7$

By S.T.

$\lim\limits_{x \to 4} f(x) = \boxed{7}$

Jan 7-8:37 AM

10) Given $f(x) = \begin{cases} \sqrt{-x} & \text{if } x<0 \\ 3-x & \text{if } 0 \le x < 3 \\ (x-3)^2 & \text{if } x \ge 3 \end{cases}$

$\sqrt{-x}$ | $3-x$ | $(x-3)^2$; 0, $\to 3$

Find

a) $\lim\limits_{x \to 0^-} f(x) = \sqrt{-0} = \boxed{0}$

b) $\lim\limits_{x \to 0^+} f(x) = 3-0 = \boxed{3}$

c) $\lim\limits_{x \to 0} f(x) =$ D.N.E.

d) $f(0) = 3-0 = \boxed{3}$

e) $\lim\limits_{x \to 3^-} f(x) = 3-3 = \boxed{0}$

f) $\lim\limits_{x \to 3^+} f(x) = (3-3)^2 = \boxed{0}$

g) $\lim\limits_{x \to 3} f(x) = \boxed{0}$

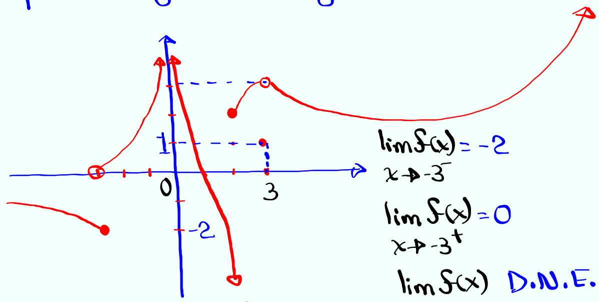
h) $f(3) = (3-3)^2 = \boxed{0}$

i) Discuss continuity at $x=0$. Not Cont. $\lim\limits_{x \to 0} f(x) \ne f(0)$

j) Discuss continuity at $x=3$. Yes $\lim\limits_{x \to 3} f(x) = f(3)$

Jan 7-8:51 AM

Graph of $y=f(x)$ is given below



$\lim_{x\to -3^-} f(x) = -2$

$\lim_{x\to -3^+} f(x) = 0$

$\lim_{x\to -3} f(x)$ D.N.E.

$f(-3) = -2$

$\lim_{x\to 0^-} f(x) = \boxed{\infty}$ $\lim_{x\to 0^+} f(x) = \boxed{\infty}$

$\lim_{x\to 0} f(x) = \boxed{\infty}$ $f(0)$ undefined

$\lim_{x\to 3^-} f(x) = \boxed{3}$ $\lim_{x\to 3^+} f(x) = \boxed{3}$ $\lim_{x\to 3} f(x) = \boxed{3}$

$F(3) = 1$

Is $f(x)$ cont. at $x=3$? Discuss

NO

$\lim_{x\to 3} f(x) \neq f(3)$

$3 \neq 1$

Jan 7-9:07 AM

Class QZ 4

1) Evaluate $\lim_{x\to -3} \frac{x^2-9}{x^2+2x-3} = \frac{(-3)^2-9}{(-3)^2+2(-3)-3} = \frac{9-9}{9-6-3} = \frac{0}{0}$ I.F.

$= \lim_{x\to -3} \frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(x-1)} = \lim_{x\to -3} \frac{x-3}{x-1} = \frac{-3-3}{-3-1} = \frac{-6}{-4} = \boxed{\frac{3}{2}}$

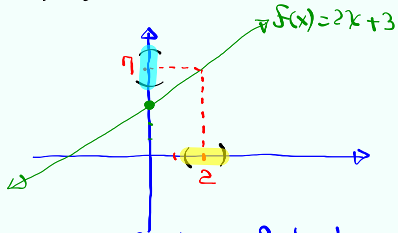
2) Evaluate $\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = \frac{\frac{1}{1}-1}{1-1} = \frac{1-1}{1-1} = \frac{0}{0}$ I.F.

LCD $= x$

$= \lim_{x\to 1} \frac{x\left(\frac{1}{x}-1\right)}{x(x-1)} = \lim_{x\to 1} \frac{\overset{-1}{\cancel{1-x}}}{x\cancel{(x-1)}} = \lim_{x\to 1} \frac{-1}{x} = \frac{-1}{1} = \boxed{-1}$

Jan 7-9:19 AM

$$\lim_{x\to 2}(2x+3) = 2(2)+3 = \boxed{7}$$



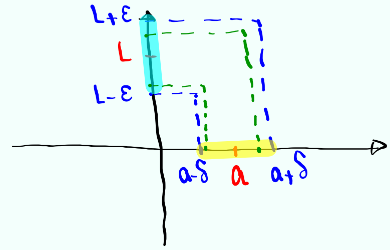
Exact Definition of limit:

For every $\varepsilon > 0$, there is a $\delta > 0$ such that

Epsilon

Delta

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad |x-a| < \delta$$



Jan 7-9:58 AM

Find ε and δ such that $\lim_{x\to 2}(2x+3) = 7$.

$f(x) = 2x+3$

$a = 2$

$L = 7$ ✓

1) verify the limit

$$\lim_{x\to 2}(2x+3) = 2(2)+3$$
$$= 4+3$$
$$= 7 \checkmark$$

2) For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad |x-a| < \delta$$

$$|2x+3-7| < \varepsilon \quad \text{whenever} \quad |x-2| < \delta$$

$$|2x-4| < \varepsilon$$

$$|2(x-2)| < \varepsilon$$

$|ab| = |a||b|$ $\quad |2|\,|x-2| < \varepsilon$

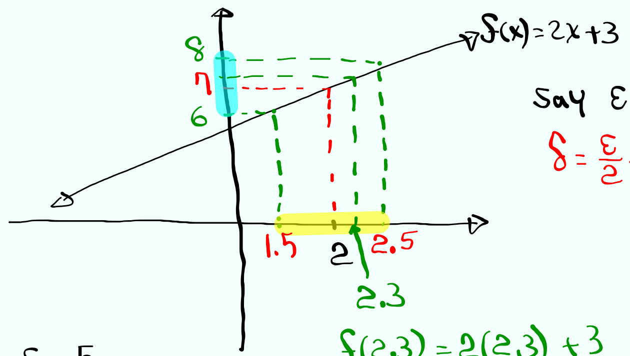
$$2|x-2| < \varepsilon$$

Divide by 2

$$|x-2| < \frac{\varepsilon}{2}$$

$$\boxed{\text{Pick } \delta = \frac{\varepsilon}{2}}$$

Jan 7-10:04 AM

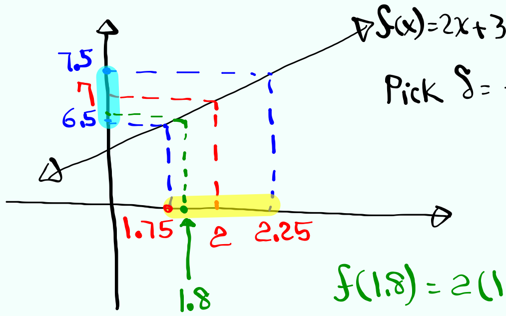


Say $\varepsilon = 1$

$\delta = \frac{\varepsilon}{2} = \frac{1}{2}$

Say $\varepsilon = .5$

$f(2.3) = 2(2.3) + 3$
$= 4.6 + 3 = \boxed{7.6}$



Pick $\delta = \frac{\varepsilon}{2} = \frac{.5}{2} = .25$

$f(1.8) = 2(1.8) + 3 = 3.6 + 3 = \boxed{6.6}$

Jan 7-10:12 AM

Find ε and δ to prove $\lim_{x \to 10} \left(\frac{1}{2}x - 4\right) = 1$

1) verify $\lim_{x \to 10} \left(\frac{1}{2}x - 4\right) = 1$

$= \frac{1}{2}(10) - 4$

$= 5 - 4 = 1$ ✓

$f(x) = \frac{1}{2}x - 4$

$a = 10$

$L = 1$ ✓

2) For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$\left|\frac{1}{2}x - 4 - 1\right| < \varepsilon$ whenever $|x - 10| < \delta$

$\left|\frac{1}{2}x - 5\right| < \varepsilon$

$\left|\frac{x}{2} - \frac{10}{2}\right| < \varepsilon$

$\left|\frac{x - 10}{2}\right| < \varepsilon$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$\frac{|x - 10|}{|2|} < \varepsilon$

$\frac{|x-10|}{2} < \varepsilon$

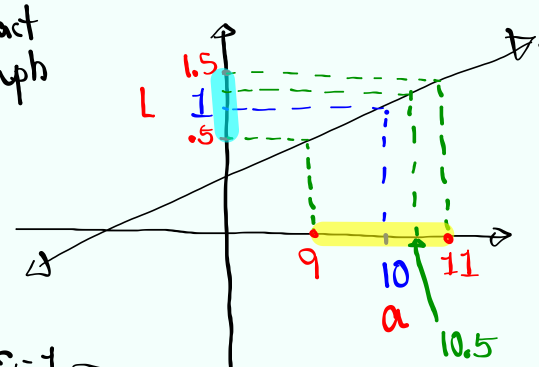
multiply by 2

$|x - 10| < 2\varepsilon$

$\boxed{\text{Pick } \delta = 2\varepsilon}$

Jan 7-10:19 AM

Not exact graph

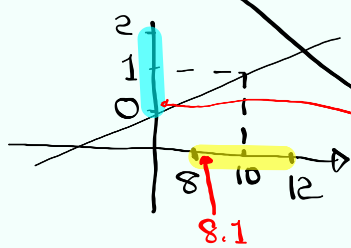


Say $\varepsilon = .5$

$\delta = 2\varepsilon$
$= 2(.5) = 1$

$f(10.5) = \frac{1}{2}(10.5) - 4$
$= 5.25 - 4 = \boxed{1.25}$

If $\varepsilon = 1$



$\delta = 2\varepsilon = 2(1) = 2$

$f(8.1) = \frac{1}{2}(8.1) - 4 = 4.05 - 4 = \boxed{.05}$

Jan 7-10:29 AM

For $\varepsilon > 0$, find a $\delta > 0$ such that $\lim_{x\to 6}\left(\frac{2}{3}x+8\right) = 12$. ✓

1) verify $\lim_{x\to 6}\left(\frac{2}{3}x+8\right) = 12$

$\lim_{x\to 6}\left(\frac{2}{3}x+8\right) = \frac{2}{\cancel{3}}(\cancel{6}^{\,2}) + 8$
$= 4 + 8 = 12$ ✓

$\boxed{\begin{array}{l} f(x) = \frac{2}{3}x+8 \\ a = 6 \\ L = 12 \checkmark \end{array}}$

2) For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$\left|\frac{2}{3}x + 8 - 12\right| < \varepsilon$ whenever $|x - 6| < \delta$

$\left|\frac{2}{3}x - 4\right| < \varepsilon$

Multiply by 3

$3\left|\frac{2}{3}x - 4\right| < 3\varepsilon$

$|2x - 12| < 3\varepsilon$

Divide by 2

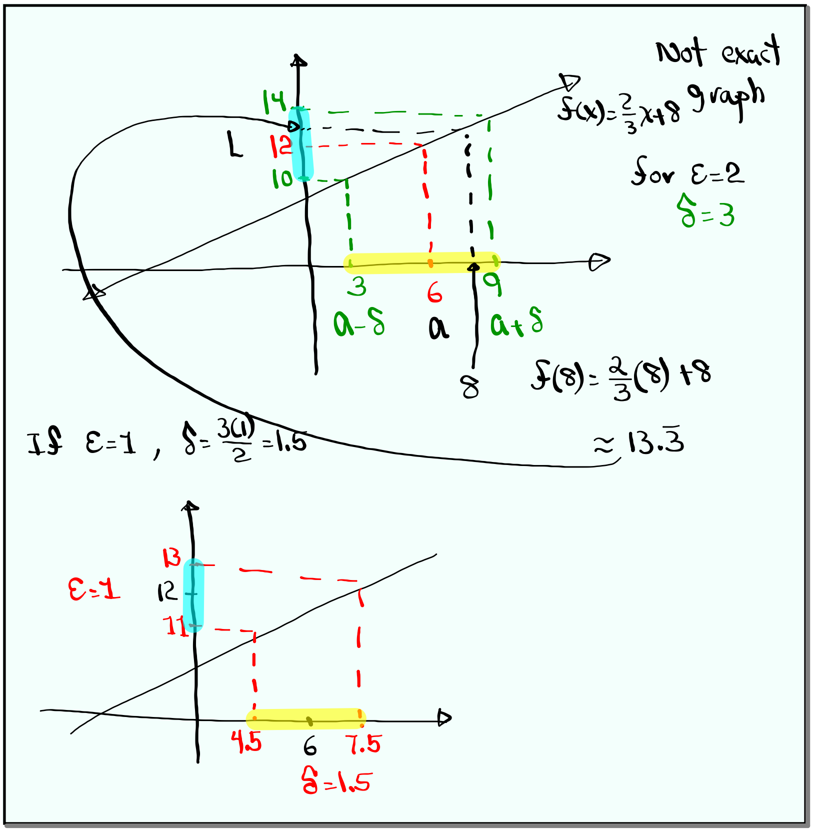
$\frac{1}{2}|2x - 12| < \frac{1}{2}\cdot 3\varepsilon$

$\to |x - 6| < \frac{3\varepsilon}{2}$

Pick $\boxed{\delta = \frac{3\varepsilon}{2}}$

If $\varepsilon = 2 \to \delta = 3$

Jan 7-10:37 AM



Jan 7-10:49 AM

For $\boxed{\varepsilon = 1}$, find a $\delta > 0$ such that $\lim_{x \to 3} x^2 = 9$.

For $\varepsilon = 1$, find $\delta > 0$ such that

$f(x) = x^2$

$a = 3$

$L = 9$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

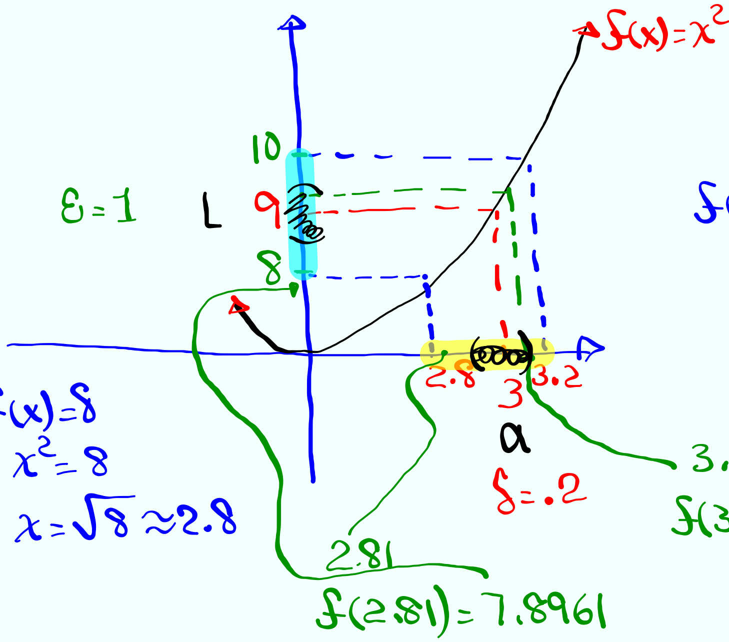
$|x^2 - 9| < 1$ whenever $|x - 3| < \delta$

$|(x+3)(x-3)| < 1$

$|x+3|\,|x-3| < 1$

↑ we must do something about it.

Jan 7-10:56 AM



Not exact Graph

$f(x)=10$
$x^2=10$
$x=\sqrt{10}$
$x\approx 3.2$

$f(x)=8$
$x^2=8$
$x=\sqrt{8}\approx 2.8$

Jan 7-11:02 AM

For $\boxed{\varepsilon = 1}$, find a $\delta>0$ such that $\lim_{x\to 3} x^2 = 9$.

For $\varepsilon=1$, find $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$f(x)=x^2$
$a=3$
$L=9$ ✓

$|x^2-9|<1$ whenever $|x-3|<\delta$

$|(x+3)(x-3)|<1$

$|x+3|\,|x-3|<1$

we must do something about it.
we must bound it

If we wish $\delta \le 1 \rightarrow$ $|x-3|<1$

$-1<x-3<1$

Add 6 to all sides

$-1+6<x-3+6<1+6$

$-7<5<x+3<7$

$-7<x+3<7$

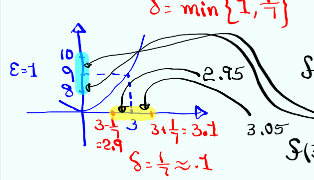
$|x+3|<7$

$|x+3|\,|x-3|<7|x-3|<1$

$|x-3|<\frac{1}{7}$

$\delta=\frac{1}{7}$

$\delta=\min\{1,\frac{1}{7}\}$

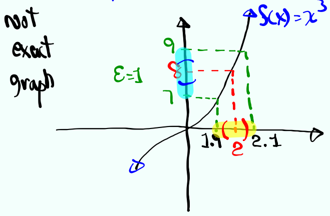


$f(2.95)=2.95^2$
$=8.7025$

$f(3.05)=3.05^2=9.3025$

Jan 7-10:56 AM

For $\varepsilon = 1$, find $\delta > 0$ such that $\lim_{x \to 2} x^3 = 8$



$f(x) = x^3$

$a = 2$

$L = 8$ ✓

$x^3 = 9 \qquad x = \sqrt[3]{9} \approx 2.1$

$x^3 = 7 \qquad x = \sqrt[3]{7} \approx 1.9$

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^3 - 8| < \varepsilon$ " $|x - 2| < \delta$

$|(x-2)(x^2+2x+4)| < \varepsilon \qquad |x-2| < \delta$

If we wish $\delta \le 1$

$|x - 2| < 1$

$-1 < x - 2 < 1$

$1 < x < 3$

If $x < 3$

$x^2 < 9$

$2x < 6$

$x^2 + 2x + 4 < 9 + 6 + 4$

$x^2 + 2x + 4 < 19$

$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

$x^3 - 2^3 = (x-2)(x^2 + 2x + 4)$

$\underbrace{|x-2|}_{\text{Keep}} \underbrace{|x^2 + 2x + 4|}_{\text{Bound}} < \varepsilon$

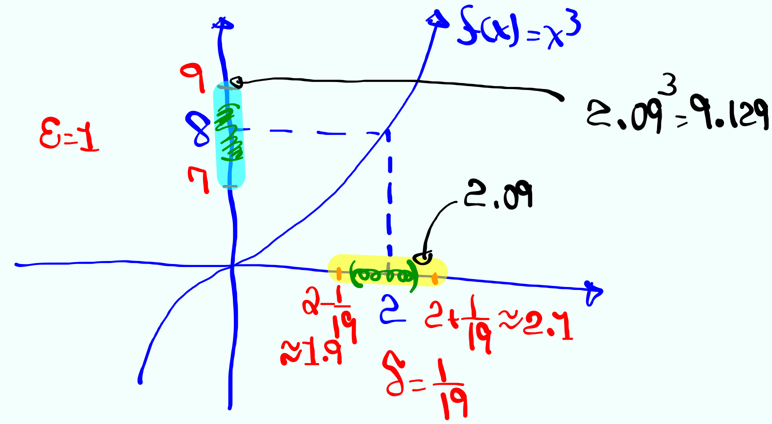
$|x-2| \cdot 19 < \varepsilon$

$|x-2| < \frac{\varepsilon}{19}$

$\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$

Jan 7-11:20 AM

Say $\varepsilon = 1$, $\delta = \min\left\{1, \frac{1}{19}\right\} = \frac{1}{19}$



Jan 7-11:35 AM

For $\varepsilon>0$, find $\delta>0$ such that $\lim_{x\to 2}(x^2-3)=1$ ✓

$f(x)=x^2-3$

$a=2$

$L=1$ ✓

For every $\varepsilon>0$, there is a $\delta>0$ such that

$$|f(x)-L|<\varepsilon \quad \text{whenever} \quad |x-a|<\delta$$

$$|x^2-3-1|<\varepsilon \quad \text{"} \quad |x-2|<\delta$$

$$|x^2-4|<\varepsilon \quad \text{"} \quad |x-2|<\delta$$

$$|(x+2)(x-2)|<\varepsilon \quad \text{"} \quad |x-2|<\delta$$

$$\underbrace{|x+2|}_{\text{Bound}}\,\underbrace{|x-2|}_{\text{Keep}}<\varepsilon \quad \text{"} \quad |x-2|<\delta$$

$$c\,|x-2|<\varepsilon \rightarrow |x-2|<\frac{\varepsilon}{c}$$

$$\delta=\min\left\{1,\frac{\varepsilon}{c}\right\}$$

If $\boxed{\delta\le 1}$, $|x-2|<1$

$-1<x-2<1$

Add 4

$3<x+2<5$

$|x+2|<5$

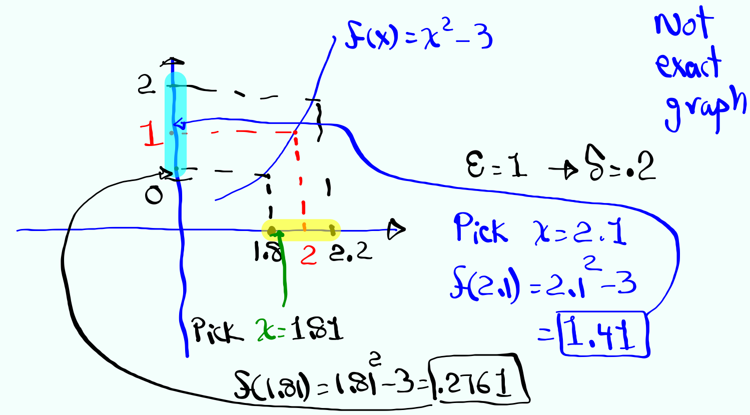
$$\delta=\min\left\{1,\frac{\varepsilon}{5}\right\}$$

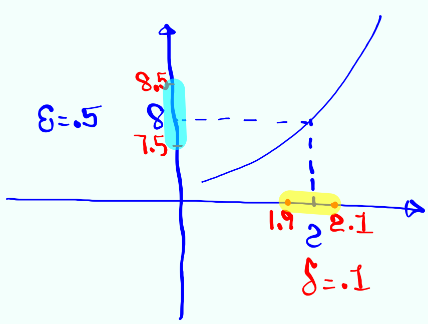
for $\varepsilon=1$, $\delta=\min\left\{1,\frac{1}{5}\right\}$

$=.2$

$\frac{1}{5}=.2$

Jan 7-11:38 AM





$$\delta=\min\left\{1,\frac{\varepsilon}{5}\right\}$$

$$=\min\left\{1,\frac{.5}{5}\right\}$$

$$=\min\{1,.1\}$$

$$=.1$$

Jan 7-11:47 AM

How to Solve $|x| < K$, $K > 0$

Answer $-K < x < K$

Solve $|x| < 5$ Ans. $-5 < x < 5$

Solve $|x-3| < 4$

$-4 < x-3 < 4$

Add 3

$-4+3 < x-3+3 < 4+3$

$-1 < x < 7$

Solve $|2x+6| < 10$

$-10 < 2x+6 < 10$

Subtract 6

$-16 < 2x < 4$

Divide by 2

$\boxed{-8 < x < 2}$

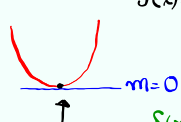
Jan 7-11:56 AM

Find a point on the graph of $f(x) = x^2 - 2x$ where we have horizontal tangent line.

No Calculus:

$f(x) = x^2 - 2x$

Parabola opens upward



vertex

$(1,-1)$ ✓

$f(x) = x^2 - 2x + 1 - 1$

$= (x-1)^2 - 1$

$f(x) = a(x-h)^2 + K$

Vertex (h, k)

With Calculus:

$$m_{tan.line} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) - x^2 + 2x}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-2x} - 2h \cancel{-x^2} \cancel{+2x}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x + h - 2)}{\cancel{h}}$$

$= 2x - 2$

Wait a minute

horizontal lines have zero slope

$2x - 2 = 0$

$2x = 2$

$\boxed{x = 1}$

Point $(1, f(1)) =$

$(1, -1)$

Jan 7-12:00 PM

Class QZ 5:

Use Quadratic Formula to Solve $x^2+25=10x$.

$$x^2+25=10x$$

$$x^2+25-10x=0$$

$$x^2-10x+25=0$$

$a=1 \quad b=-10 \quad c=25$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(25)}}{2(1)}$$

$$=\frac{10\pm\sqrt{100-100}}{2}=\frac{10\pm 0}{2}=\frac{10}{2}=\boxed{5}$$

Jan 7-12:11 PM